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On a Compromise Solution for Solving Multi-Objective Convex Programming Problems

Alia Youssef Gebreel

Abstract— This paper presents an alternate method to find an evenly efficient solution for all weights of multi-objective convex programming problems with conflicting objectives. The main idea behind the proposed methodology is to combine the attractive features of both the hybrid method and normal- boundary intersection method. This approach is called Alia's method, and its solution is expressed as Alia point. This point is the best efficient point or very close to it in the efficient front.

Index Terms— Multi- objective optimization problem (MOP); efficient solution; utopia point; Normal- Boundary Intersection (NBI); and the hybrid method (HP).

1 Introduction

A multi- objective (or multicriteria, or multi-performance) optimization problem has a number of objective functions which are to be minimized or maximized. If all objective functions and constraint functions are linear, the resulting multi- objective optimization problem (MOP) is called a multi- objective linear program (MOLP). However, if any of the objective or constraint functions are nonlinear, the resulting problem is called a nonlinear multi- objective problem.

Multi- objective optimization is sometimes referred to as vector optimization problem (VOP), because a vector of objectives, instead of a single objective, is optimized [10, 11].

The term vector optimization is sometimes used to denote the problem of identifying the efficient set. However, this is not always enough. We want to obtain only one solution; this means that we must find a way to put the efficient solutions in a complete order.

In general, multi-objective optimization problems are solved by scalarization. Scalarization means converting the problem into a single or a family of single objective optimization problems with a real-valued objective function, termed the scalarizing function, depending possibly on some parameters [10]. These objectives are usually in incommensurate and conflicting with one another, there normally exist:

1- Infinite number of efficient (non- dominated, Pareto- optimal, or non-inferior) solutions in the MOPs. The problem is how to search for a best compromise solution with these multiple objectives being considered simultaneously.

2- Not all objectives can simultaneously arrive at their optimal levels. So, an assumed utility function is used to choose appropriate solutions [2, 8, 18].

Over the past years, many researches provide different approaches to improve the ability of extracting efficient solutions.

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Generally, gradient-based methods for solving multi-objective optimization problems (e.g., weighted method, epsilon constraint method, and Normal Boundary Intersection (NBI)) require solving at least one single-objective optimization problem for each Pareto optimal point, and thus solving many problems to find the Pareto frontier. These methods can be computationally expensive with an increase in the number of variables and/or constraints of the optimization problem [16].

When this work focuses on improving the NBI method, the elicitation of an efficient solution in the efficient set is given. So, this research deals with a novel methodology for practical implementation of convex multicriteria optimization problems. It combines the hybrid method with NBI by a new direction of search. Where, both methods have strong and weak points:

- Gradient-based methods have high convergence to local a Pareto front, but a low ability to find the global Pareto frontier and disjoint parts of a Pareto frontier;
- Hybrid method has a strong ability to find the global optimal solutions, but it may be difficult to specify the parameter values for the objective functions [9, 17].

It is clear that creating such an optimization methodology which combines the strong points of both approaches is the more efficient solution.

This paper presents the fundamental theories, the method, and examples that illustrate the favorable efficient solution for the decision maker.

The rest of this paper is organized as follows. Problem formulation is presented in section 2. Basic concepts are treated in section 3. In section 4, the proposed method for solving the multi- objective convex programming problems is described in detail. To demonstrate the performance of this method, some examples involving linear and non- linear convex MOPs have been simulated in section 5. Finally, section 6 concludes this research.

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2 Problem Formulation

Consider the following multi-objective nonlinear programming problem:

(MOP): Min F(x) = ($f_1(\mathbf{x}),\ f_2(\mathbf{x}), \ldots\ldots, f_k(\mathbf{x})$ **)** , $k \geq 2$,

Subject to

$M = \{\mathbf{x} \in R^n \,/\, g_r(\mathbf{x}) \leq 0,\ r = 1,2,\ldots..,m\}$.

Where, $f_i(\mathbf{x})$, $g_r(\mathbf{x})$, $i = 1,2,\ldots..,k$, $r = 1,2,\ldots..,m$ are continuous functions of class $C^{(1)}$ on R^n (the first order partial differential exists and continuous).

The set M is assumed to satisify Slater constraint qualification (or in other constraint qualification) [12, 15].

Assume that

$\mathbf{f_i(x_i^*) = \min f_i(x),\ \ i = 1,2,\ldots, k,}$

$\mathbf{x \in M.}$

3 Basic Concepts and Definitions

Let us introduce some terminology:

3.1 Efficient Solution

A decision vector $x^* \in S$ is said to be an ***efficient solution*** if there does not exist another decision vector $x \in S$ such that $f_i(x) \leq f_i(x^*)$ for all $i = 1, 2, \ldots\ldots, k$ and $f_j(x) < f_j(x^*)$ for at least one index j.

3.2 Efficient Front

The collection of all Pareto- optimal solutions is called the Pareto- optimal set. The image of the Pareto- optimal set by F is referred as the ***Pareto- optimal front*** **(*efficient frontier* or *tradeoff surface*)** [9, 14].

3.3 Utopia Point

The point $(f_1(\mathbf{x}_1^*), f_2(\mathbf{x}_2^*), \ldots, f_k(\mathbf{x}_k^*))$ in the objective space is called the ***Utopia point***.

3.4 Utopia Line

The line joining two optimal points in bi-objective case is called the ***Utopia line*** [1].

3.5 Utopia Hyperplane

The plane passing through the points $f(\mathbf{x}_i^*)$, all $i = 1, 2, \ldots, k$ in the objective space is called the ***Utopia hyperplane***.

3.6 Utopia Hypersphere

A sphere of center the utopia point and any positive radius in the objective space is called a ***Utopia hypersphere***.

3.7 Alia Line

The line passing through the utopia point and perpendicular to the utopia hyperplane is called the ***A line***. The figure (1) illustrates this definition.

3.8 Alia Point

The point of intersection of the ***A line*** and the efficient front is called the ***A point***.

3.9 Best Efficient Point

The best efficient point on the efficient front that has the shortest distance from the utopia point is called the ***B point***. It is clear that, it is the point of common adjacent between the efficient front and a utopia hypersphere. In this case, the radius is the best.

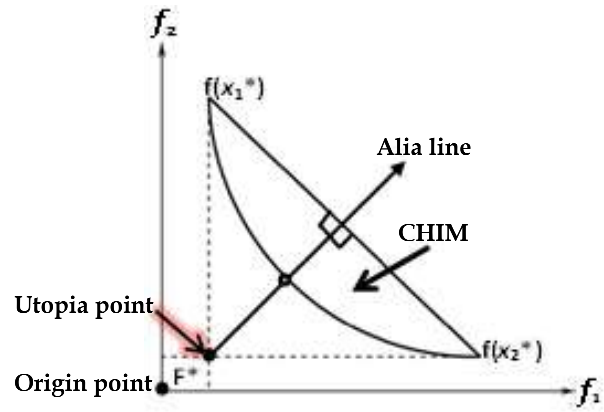

Fig.(1) The conflicting two optimal objective vectors and Alia line in a multi- objective problem.

3.10 Distance of Alia Point (dA)

Given ***A point*** and the utopia point, the distance (d) between these two points is given as follows:

$dA \equiv d$ (***A point***, utopia point). **(1)**

3.11 Distance of the Best Efficient Point (dB)

Given ***B point*** and the utopia point, the distance between these two points is given as follows:

$dB \equiv d$ (***B point***, utopia point). **(2)**

3.12 Normal Boundary Intersection Method

Das and Dennis (1998) proposed the Normal Boundary Intersection (NBI) method where a series of single- objective optimizations is solved on normal lines to the utopia line. The NBI method gives fairly uniform solution and can treat problem with non- convex region on the Pareto front. It achieves this by imposing equality constraints along equally spaced lines or hyperplanes in multidimensional case [7].

NBI is a two- step method:

1- For all individual objective f_i, $i \in \{1, 2, \ldots, k\}$ the respective global minimizes $x_i^* \in R$ are determined.

2- The Convex Hull of the Individual Minima in the objective space called CHIM, i.e., the convex hull of the vectors $\{\mathbf{f}(x_1^*), \ldots, \mathbf{f}(x_k^*)\}$ can be expressed by means of the matrix $\Phi = (\mathbf{f}\ (x_1^*),\ \mathbf{f}\ (x_2^*), \ldots, \mathbf{f}\ (x_k^*)) \subset R^k, \in R^{k \times k}$ as $\{\Phi\mathbf{w} / \mathbf{w} \in R^k, \sum_{i=1}^{k} w_i = 1,\ w_i \geq 0\}$. Where, $\Phi\mathbf{w}$ denotes the starting point on the CHIM-simplex. By varying the weight vector w, i.e; by varying the starting point on the CHIM- simplex and by solving the resulting NBI substituting problems a subset of the efficient set can be generated.

Optimization Problem (NBI Substituting Problem):

(NBI): Min δ (with the additional constraint)

Subject to

$\Phi \mathbf{w} + \mathbf{N}.\, \delta = F(\mathbf{x})$,
$\mathbf{x} \in S$

The following figure illustrates the CHIM- simplex in the bicriteria case.

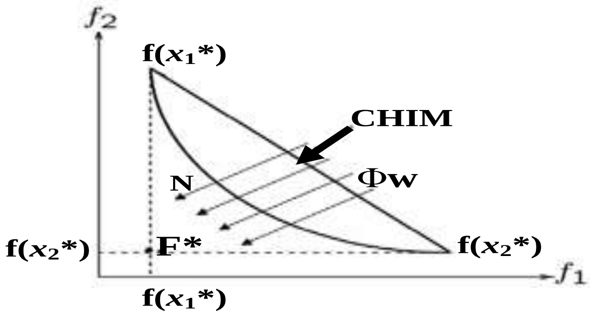

Fig. (2) Pareto- optimal front for a two- objective problem.

Where, **N** is the normal to the CHIM- simplex pointing towards the origin, and $\delta \in R$ represents the set of points on that normal. Then the point of intersection between the normal and the boundary of objective functions closest to the origin is the solution of the problem [3, 4, 5, 6, 16].

3-13 Payoff Table:

A payoff table can be formed as shown in Table (1). It constructs by using the decision vectors obtained when calculating the ideal objective vector. Column i of the payoff table displays the values of all the objective functions calculated at the point where f_i obtained its minimal value. Hence, z_i^* is at the main diagonal of the table. The maximal value of the row i in the payoff table can be selected as an estimate of the upper bound of the objective f_i for i =1 ,… ..., k over the Pareto optimal set [5, 9].

Table (1) Pay- off table

Data	(x_1^*)	(x_2^*)	… ...	(x_i^*)
f_1	$f_1(x_1^*)$	$f_1(x_2^*)$	… ...	$f_1(x_i^*)$
f_2	$f_2(x_1^*)$	$f_2(x_2^*)$	… ...	$f_2(x_i^*)$
.	.	.	.	.
.	.	.	.	.
f_i	$f_i(x_1^*)$	$f_i(x_2^*)$	… ...	$f_i(x_i^*)$

3-14 Hybrid method:

It is well known that the efficient solutions of (MOP) could be generated via the hybrid approach repesented by the following problem [9]:

(HP): Min $\sum_{i=1}^{k} w_i f_i(\mathbf{x})$

Subject to

$f_i(\mathbf{x}) \le \varepsilon_i$ for all i = 1, 2, ..., k,

$\mathbf{x} \in M$

where, $w_i > 0$, i= 1, 2, ..., k, $\sum_{i=1}^{k} w_i = 1$, and $\varepsilon_i \in R$, i = 1, 2, ... , k which are chosen such that problem (HP) is feasible.

Problems (MOP), (HP) are related to each other by the following theorem [9, 18].

Theorem 1:

A point $\bar{x} \in M$ is an efficient solution of (HOP) iff $\bar{x}$ is an optimal solution of problem (HP) for a certain $w_i > 0$, feasible ε_ℓ, i, ℓ =1, 2, ... , k.

4 Alia's Method

Alia's method is produced to solve multi-objective convex programming problems for all weights with conflicting objectives. It can be regarded as a general case of the value function for providing only one efficient solution. Its solution is called Alia point. This method combines the main positive features of the hybrid method and NBI method. It scalarizes a set of objectives and controlling variable "δ" multiplying by ($\|N\|^2$) into a single objective. On other hand, all objectives and the normalized controlling vector "Nδ" are converted into constraints by setting a lower bound to each of them in minimizing problem. The lower bound is the optimal value of each objective. It is depended on the normal of objective vectors. ***A point*** is the best point or comes very close to the ***B point*** based on its distance of the utopia point.

4-1 The steps of the proposed methodology:

The steps of this methodology are summarized as follows:

1- Calculate the individual minima (or maxima) f_i^* of the performances f_i with i = 1,2,…………., k, which are determined from solving the MOP for all individual objective f_i.

2- Construct the Pay-off matrix.

3- Determine the normal of objective vectors (N).

4- Solve the proposed formulation to obtain an efficient solution. Then, ***A point*** is the optimal solution for Alia's model for all weights, and a compromise solution for minimizing (or maximizing) the MOP.

Now, consider the following problem denoted as (AP).

(AP): Min $\left(\sum_{i=1}^{k} w_i f_i(\mathbf{x}) + \|N\|^2 \delta \right)$,

Subject to

$f_i(\mathbf{x}) - n_i \delta \le f_i^*$, i = 1, 2, 3, ... , k,

$\mathbf{x} \in M$,

Where:

$\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is a vector of the decision variables, n is the number of the decision variables.

$w_1, w_2, \ldots, w_k$ are the weights of the objective functions $f_i(x)$, $w_i > 0$, i= 1, 2, ..., k, $\sum_{i=1}^{k} w_i = 1$.

k is the No. of objective functions.

$N = (n_1, n_2, \ldots, n_k)$ is the normal vector directed in the positive direction to the utopia hyperplane.

Nδ is the normalized controlling vector.

f_i^* is the optimum value of $f_i(\mathbf{x})$ over M, i = 1, 2, 3,…., k,

δ is a real variable; variable is clearly positive due to the feasibility of the constraints.

It is clear that the constraints of problem (AP) satisfy the Slater constraints qualification.

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Formulating the Kuhn- Tucker (K. T) conditions for problems (HP), and (AP), the relation between the two problems will be clear:

(1) Kuhn- Tucker conditions for problem (HP):

$$\sum_{i=1}^{k} w_i \frac{\partial f_i(x)}{\partial x_j} + \sum_{i=1}^{k} u_i \frac{\partial f_i(x)}{\partial x_j} + \sum_{r=1}^{m} v_r \frac{\partial g_r(x)}{\partial x_j} = 0,$$

$j = 1, 2, 3, \ldots, n,$ **(3)**

$u_i (f_i(x) - \varepsilon_i) = 0, \quad i = 1,2,\ldots, k,$ **(4)**

$v_r g_r(x) = 0, \quad r = 1,2,\ldots,m,$ **(5)**

$f_i(x) \leq \varepsilon_i, \quad i = 1,2,\ldots, k,$ **(6)**

$g_r(x) \leq 0, \quad r = 1,2,\ldots,m,$ **(7)**

$u_i \geq 0, \quad i = 1,2,\ldots, k,$ **(8)**

$v_r \geq 0, \quad r = 1,2,\ldots,m,$ **(9)**

$\sum_{i=1}^{k} w_i = 1, \ w_i > 0, \quad i = 1, 2,\ldots, k.$ **(10)**

(2) Kuhn- Tucker conditions for problem (AP)

$$\sum_{i=1}^{k} w_i \frac{\partial f_i(x)}{\partial x_j} + \sum_{i=1}^{k} \mu_i \frac{\partial f_i(x)}{\partial x_j} + \sum_{r=1}^{m} \alpha_r \frac{\partial g_r(x)}{\partial x_j} = 0,$$

$j = 1, 2, 3, \ldots, n,$ **(11)**

$\sum_{i=1}^{k} n_i \mu_i = \|N\|^2,$ **(12)**

$f_i(\mathbf{x}) - n_i \delta \leq f_i^*, \quad i = 1, 2, 3, \ldots, k,$ **(13)**

$g_r(\mathbf{x}) \leq 0, \quad r = 1,2,\ldots,m,$ **(14)**

$\mu_i (f_i(\mathbf{x}) - n_i \delta - f_i^*) = 0, \quad i = 1, 2, 3, \ldots, k,$ **(15)**

$\alpha_r g_r(\mathbf{x}) = 0, \quad r = 1,2,\ldots,m,$ **(16)**

$\mu_i \geq 0, \quad i = 1, 2, 3, \ldots, k,$ **(17)**

$\alpha_r \geq 0, \quad r = 1,2,\ldots,m,$ **(18)**

Lemma 1:

If for $\overline{w} = (\overline{w}_1, \overline{w}_2, \ldots, \overline{w}_k) > 0$, $(\overline{x}, \overline{\delta}, \overline{\mu}, \overline{\alpha})$ satisfies the Kuhn- Tucker conditions (2), then for $\overline{w} > 0$, there exists a feasible $\overline{\varepsilon} = (\overline{\varepsilon}_1, \overline{\varepsilon}_2, \ldots, \overline{\varepsilon}_k)$, such that $\exists\ (\overline{x}, \overline{u}, \overline{v})$ satisfies the Kuhn- Tucker conditions (1).

Proof:

Let for $\overline{w} > 0$, the point $(\overline{x}, \overline{\delta}, \overline{\mu}, \overline{\alpha})$ satisfies the Kuhn- Tucker conditions (2), then it is clear from the formulation of the Kuhn- Tucker conditions for problems (HP), and (AP) that if we take $\overline{\varepsilon}_i = n_i \overline{\delta} - f_i^*$, $i = 1, 2, 3, \ldots, k$, then $\overline{\varepsilon}$ will be feasible for Kuhn- Tucker conditions (1). Also, it evident that $\overline{u} = \overline{\mu}$, $\overline{v} = \overline{\alpha}$, and hence $(\overline{x}, \overline{\mu}, \overline{\alpha})$ satisfies the Kuhn- Tucker conditions (1).

Lemma 2:

Let the constraints of problem (AP) satisfy Slater constraints qualification (on any other constraints qualification) [12]. If for $\overline{w} > 0$, $(\overline{x}, \overline{\delta})$ is an optimal solution of problem (AP), then $\overline{x}$ is an efficient solution of problem (MOP).

Proof:

Let $(\overline{x}, \overline{\delta})$ be an optimal solution of this problem for $\overline{w} > 0$. Then from the necessary optimality theorem of nonlinear problem [12, 13], it follows that $\exists\ \overline{\mu}, \overline{\alpha} > 0$ such that $(\overline{x}, \overline{\delta}, \overline{\mu}, \overline{\alpha})$ satisfy the Kuhn- Tucker conditions (2) and hence by Lemma 1, $(\overline{x}, \overline{\mu}, \overline{\alpha})$ satisfy the Kuhn- Tucker conditions (1) for $\overline{w} > 0$, and feasible $\overline{\varepsilon}$.

From the sufficient optimality theorem of nonlinear problem [12, 13], it follows that $\overline{x}$ is an optimal solution of problem (HP) for $\overline{w} > 0$, and feasible $\overline{\varepsilon}$. Hence from theorem 1, we deduce that $\overline{x}$ is an efficient solution of problem (MOP).

4-2 Existence of Alia point:

Alia point must pass the normal and the convex hull of individual minima (CHIM). As shown in Fig. (3), this point is indicated for two dimensions problem. Where, it is located at the intersection of ***A line*** "pointing from the utopia point towards the utopia line in 2D" and CHIM.

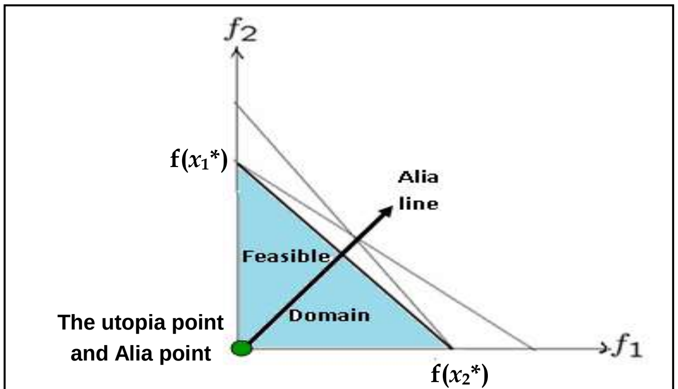

Fig. (3- a)

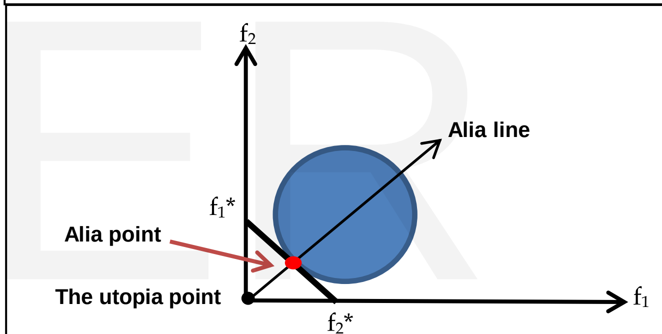

Fig. (3- b)

Fig. (3) A graphical representation of Alia point for bi- objective problem in non-conflicting and conflicting objectives, respectively.

Remark 1:

This definition shows the existence of ***A point*** in the efficient set for any (VOP), whether these objectives are conflicting or not conflicting.

Theorem 2:

If for $\overline{w} > 0$, $(\overline{x}, \overline{\delta})$ is an optimal solution of problem (AP) such that $f_i(\overline{x}) - n_i \overline{\delta} = f_i^*$, $i = 1, 2, 3, \ldots, k$, then $\overline{x}$ will be an *Alia efficient point* for problem (MOP).

Proof:

Let for $\overline{w} > 0$, $(\overline{x}, \overline{\delta})$ be an optimal solution of problem (AP). Then from Lemma 2, it follows that $\overline{x}$ is an efficient solution of problem (AP). Since, $\overline{x}$ satisfies the relation $f_i(\overline{x}) - n_i \overline{\delta} = f_i^*$, $i = 1, 2, 3, \ldots, k$, then from the definition of ***A efficient point***, it is clear that $\overline{x}$ is an ***Alia efficient point*** and in this case $d_A = \|N\|\delta$. **(19)**

4-3 Uniqueness of Alia point

Alia's method develops for supporting the decision maker to

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find ***A point***. This point is only one efficient solution regardless of the number of Pareto- optimal points.

Theorem 3:

Assume that all the objective functions of problem (MOP) are conflicting with each other, then for any w > 0, only one ***Alia point*** of problem (MOP) could be generated from problem (AP).

Proof:

Utilizing the result of theorem 2 and assume that for another $w^* > 0$, (x^*, δ^*) is an optimal solution of problem (AP) such that

$$f_i(x^*) - n_i \delta^* = f_i^*, \; i = 1, 2, 3, \dots, k. \quad \textbf{(20)}$$

Then, x^* is an efficient solution of problem (MOP), and we have either one of the following two cases.

(1) $\delta^* \le \bar{\delta}$:

In this case $f_i(x^*) \le f_i(\bar{x})$, i = 1, 2, 3, ... , k, and this contradicts the efficiency of $\bar{x}$.

(2) $\delta^* > \bar{\delta}$:

In this case $f_i(\bar{x}) < f_i(x^*)$, i = 1, 2, 3, ... , k, and this contradicts the efficiency of x^*.

Therefore, for any w > 0, there is only one optimal solution of problem (AP) satisfying the condition $f_i(x^*) - n_i \delta = f_i^*$, i = 1, 2, 3, ... , k.

This means that there is only one unique ***A point***, which could be generated from problem (AP). By this way the theorem is proved.

The relation between the optimality of (AP) and efficiency of (MOP) problems can be illustrated as shown in the following figure.

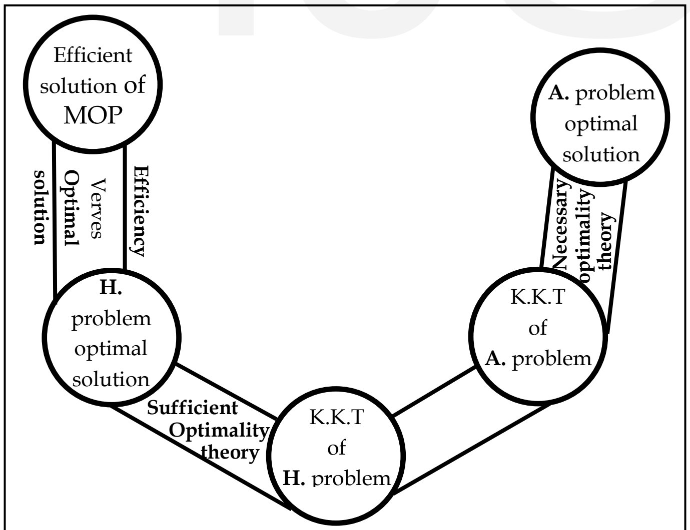

Fig. (4) The relation between the optimality of (AP) and efficiency of (MOP) problems.

Remarks 2:

1) Assume that ℓ of the objective functions $f_i(x)$, i = 1, 2, 3, ... , k. Say $F_{k-\ell+1}, F_{k-\ell+2}, \dots, F_k$ are nonconflicting with each other objectives then these objectives are eleminated from problem (MOP) and in this case ***A point*** may be nonunique for problem (AP).

2) For some multi- objective nonlinear programming porblems ***A point*** is the best point, and for some others ***A point*** is not the **B point**. But in such cases dA, and dB are too close to each other.

3) ***A point*** can be extracted easily when the weights are equals.

5 Some Illustrative Examples

The following linear and nonlinear MOP examples are introduced to clarify the proposed method.

Example 1:

Consider the following multiobjectives problem:

Min (x, y, -x-3y, $2x^2$ - 4y),
Subject to
x+ y ≥ 2,
-x+ y ≤ 2,
3x+ y≤ 6,
x ≥ 0, y ≥ 0.

For this example, let $f_1 = x$, $f_2 = y$,
$f_3 = -x - 3y$, $f_4 = 2x^2 - 4$, then
$f_1^* = 0$ attained at the point (0, 2),
$f_2^* = 0$ attained at the point (2, 0),
$f_3^* = -10$ attained at the point (1, 3),
$f_4^* = -10$ attained at the point (1, 3),
Fig. (5) Shows these individual optimal solutions.

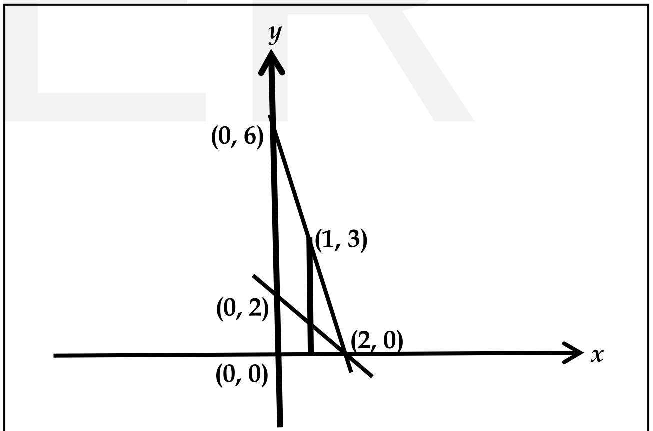

Fig. (5) The decision space of the example (1).

It is clear that f_3, f_4 are nonconflicting, then we can omitt one of them from the problem, and then N= (1, 3, 1).

If we omitt f_4, ***A point*** is the efficient solution $(\frac{10}{11}, \frac{30}{11})$.

But if we omitt f_3, ***A point*** is the efficient solution (0.3713325, 2.371332).

However, the Ad for the two points, based on three objectives, are: $d_1 = 3.015$, $d_2 = 2.527$.

Note that:

If we take N= (1, 3, 1, 1) for all four objectivesof the prolem with different weights, the resulted efficient solutions are ($\frac{10}{11}$,

$\frac{30}{11}$) and (0.76923, 2.76923). The Ad for the two points; based on four objectives together; is: d_1= 3.105, d_2= 3.02, respectively.

Example 2:

Min (x, y),

Subject to $x^2 + y^2 \le 1$.

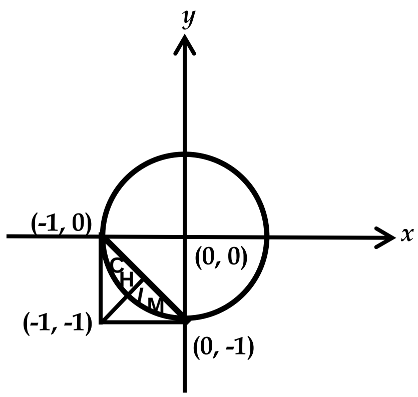

Fig.(6) The decision and objective spaces of the example (2).

As seen in Fig. (6), the normal vector N = (1, 1),

A point = ***B point*** = ($\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}$), and dA= dB=1.

Where, dA =‖N‖ δ = $\sqrt{2}$ δ

Example 3:

Min (x, y),

Subject to

$2x+ y \ge 4$, $2x+ 3y \ge 8$,

$x+ y \le 4$, $x \ge 0$, $y \ge 0$.

When this example is solved as depicted in Fig.(7), we find N= (1, 1),

A point = (1.6, 1.6), dA = 2.263,

B point = ($\frac{16}{13}, \frac{24}{13}$), dB = 2.219.

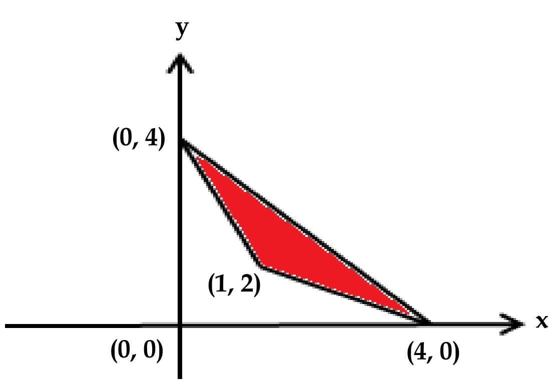

Fig.(7) The decision and objective spaces of the example (3).

Example 4:

Min ($-y + x^2$, $2y- x$),

Subject to

$x+ y \le 1$, $x \ge 0$, $y \ge 0$.

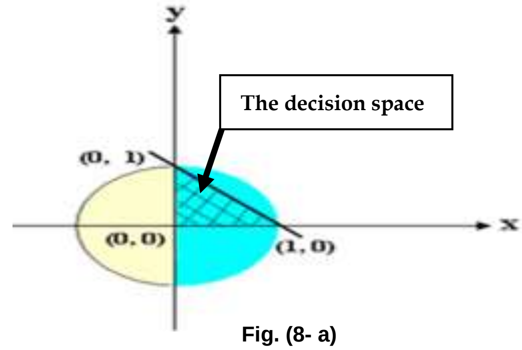

Fig. (8- a)

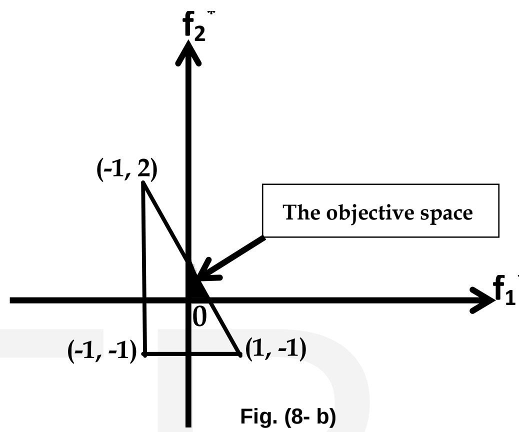

Fig. (8- b)

Fig.(8) The decision and objective spaces of the example (4).

As shown in the above figure and by solving this example using the proposed method:

N= (3, 2),

A point = (0.281, 0), dA= 1.296,

B point = (0.313, 0), dB= 1.295.

6 Conclusion

This paper presents Alia's method for solving multi-objective convex programming problems that depends on the normal of objective vectors. It yields Alia point that is an efficient point in the efficient front when all the objectives of the MOP are conflicting to each other. This point is the best efficient solution or comes very close to the best solution based on the distance of the utopia point. So, it helps the decision maker to overcome the difficulties of selecting a solution from the efficient set that he/ she wants to it. When solving MOP with equal weights, ***A point*** is as the original problem's solution or better than it.

In this method, the search's area to find the solution is less than that in other methods. Also, it can be regarded as a general case of the value function. Finally, this method of handling multiple criteria optimization problems gives a direction toward future research from a viewpoint of practical implementation.

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REFERENCES

[1] A. Messac, A. Ismail-Yahaya, C.A. Mattson, "The Normalized Normal Constraint Method for Generating the Pareto Frontier" Struct Multidisc Optim, 25, P. (86- 98), 2003.

[2] Bruce A. McCarl and Thomas H. Spreen, " Chapter XI: Multi-objective Programming", (Internet), 1997.

[3] Claus Hillermeier, "Nonlinear Multi-objective Optimization- A Generalized Homotopy Approach", Birkhäuser Verlag, Basel, Boston, Berlin, 2001.

[4] Daniel Mueller, Helmut Graeb, and Ulf Schlichtmann, "Trade-off Design of Analog Circuits Using Goal Attainment and "Wave Front" Sequential Quadratic Programming", EDAA, 2007.

[5] I. Das, "On characterizing the "Knee" of the Pareto Curve Based on Normal- Boundary Intersection", Structural Optimization, 18, P. (107-115), Springer- Verlag 1999.

[6] Indraneel Das and John Dennis, "Normal-Boundary Intersection: An Alternate Method for Generating Pareto Optimal Points in Multicriteria Optimization Problems", NASA Contractor Report 201616, ICASE Report No. 96-62, ICASE, (1996).

[7] I.Y., and O.L. de Weck "Adaptive Weighted-Sum Method for Bi-objective Optimization: Pareto Front Generation", Struct Multidisc Optim 29, P. (149-158), (2005).

[8] Jian-Bio Yang, Pratyush Sen, "Preference Modeling by Estimating Local Utility Functions for Multiobjective Optimization", European Journal of Operational Research, 95 (1996) 115- 138.

[9] Kaisa M. Miettinen, "Nonlinear Multiobjective Optimization", Kluwer Academic Publishers, 1998, and Fourth Printing 2004.

[10] Kalyanmoy Deb, "Multi-objective Optimization Using Evolutionary Algorithms", John Wiley & Sons, Ltd, 2001.

[11] Lidiane Sartini De Oliveira, and Sezimaria F.P. Saramago,"Multiobjective Optimization Techniques Applied to Engineering Problem", J. of the Braz Soc. Of Mech. Sci. & Eng., Vol. XXXll, No. 1/105, 2010.

[12] Mangasarian, O. L.,"Nonlinear Programming", McGraw- Hill, Inc., (1969), and The Society for Industrial and Applied Mathematices, (1994).

[13] Mokhtar S. Bazzraa, Hanif D. Sherali, C. M. Shetty, "Nonlinear Programming: Theory and Algorithms", John Wiley & Sons, Inc., 1979, 1993, and 2006.

[14] Ricardo P. Beausoleil, "MOSS- Multiobjective Scatter Search Applied to Non-linear Multiple Criteria Optimization", European Journal of Operational Research, P.(1-24), xxx (2004) xxx- xxx.

[15] R.T. Marler and J.S. Arora, "Survey of Multi-objective Optimization Methods for Engineering", Structural and Multidisciplinary Optimization, V. 26,N. 6, pp. (369-395), (2004).

[16] S. Siddiqui, S. Azarm, and S. A. Gabriel, " On Improving Normal Boundary Intersection Method for Generation of Pareto Frontier", Struct Multidisc Optim 46, P. (839-852), (2012).

[17] Valdimir Sevasty Anov, "Hybrid Multi- Gradient Explorer Algorithm for Global Multi-objective Optimization", American Institute of Aeronautics and Astronautics (10), 94-99, P.(1-15), eArtius, Inc. on Jan 21, 2013.

[18] Vira C. and Yacov Y. Haimes," Multiobjective Decision making: Theory and Methodology ", Elsevier Science Publishing Co., Inc., Series Volume 8, 1983.